ISRAEL JOURNAL OF MATHEMATICS, Vol. 31, Nos. 3–4, 1978

ERGODICITY OF SOME MAPPINGS OF THE CIRCLE AND THE LINE

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ABSTRACT

If ϕ is inner and has a fixed point in D, then ϕ as a mapping of the circle is exact. If ϕ has a "fixed" point on T, then the condition $\Sigma(1-|\phi_n(0)|)=\infty$ implies ϕ_m is weak mixing for all m. These results when transferred to the line by a conformal mapping of the disc onto the upper half plane give a proof of the total weak mixing for the Boole transformation.

This paper deals with the question of when a mapping of the circle is ergodic with respect to absolutely continuous measures. The question of strong mixing and weak mixing is also considered. The class of maps studied are the inner functions; that is, those ϕ which are analytic in the open unit disc D and $|\phi(e^{it})|=1$ almost everywhere. We will always mean absolute continuity and almost everywhere with respect to Lebesgue measure unless otherwise stated.

Ergodicity can be defined with respect to any quasi-invariant measure. We find a condition on an inner function ϕ which determines its ergodic character with respect to Lebesgue measure and, therefore, with respect to any absolutely continuous finite or σ-finite measure. When ϕ has a fixed point in the unit disc D we show ϕ has a Poisson measure as an invariant measure and when ϕ is not invertible ϕ is exact and , therefore, mixing of all orders. When ϕ does not have a fixed point in D, a theorem of Denjoy–Wolff [6] says ϕ has an attractive point on the circle T. In this case the condition found for ergodicity implies total ergodicity and weak mixing.

By mapping the disc D conformally onto the upper half plan, we can transfer the ergodic condition on inner functions ϕ to certain mappings of the line. This then gives a simple proof of the total ergodicity and weak mixing of the Boole transformation studied by Adler and Weiss [2]. The ergodic theory for certain

Received January 21, 1978 and in revised form May 18, 1978

other maps has been considered by Kemperman [8] and Schweiger [10]. Their results are covered by the theorems in this paper. A more comprehensive study has been made by Aaronson [1] and many of his results overlap the results in this paper.

Finally, Adler and Flatto [3] studied C^2 mappings of the circle. Their methods are different from those used in this paper; however, their results coincide with ours when the maps are finite Blaschke products.

We set down some facts about H^p spaces that we need and refer to [5] for more details.

H^p is the class of functions which are analytic in D and such that the integrals $\int_0^{2\pi}|f(re^{it})|^p dt$ are bounded for $r<1$, here $0<p<\infty$. H^∞ is the class of functions which are analytic and bounded in D. N, the Nevanlinna class, consists of those functions f which are analytic in D and such that the integrals $\int_0^{2\pi}|\log|\,|f(re^{it})|\,dt$ are bounded for $r<1$. We note that $H^p\subseteq N$ for all $0<p\leqq\infty$ and that if $f\in H^1$ then $\exp f\in N$.

The zeros of functions in N are characterized by the Blaschke condition; a sequence $z_n\in D$ is the zero set of a function in N if and only if $\Sigma(1-|z_n|)<\infty$.

Let ϕ be an inner function. We can think of $\phi(z)$ as a mapping of the circle onto itself by looking at its radial limits $\phi(e^{it})$. If ϕ is not continuous on T the unit circle, then ϕ is defined almost everywhere. If ϕ is not constant, it cannot map a set of positive measure onto a set of measure zero. In fact, Borel sets of positive measure are carried onto analytic sets of positive measure.

Suppose $f\in H^p$ and ϕ inner. Ryff [9] showed that $f(\phi(e^{it}))=f(\phi)(e^{it})$ almost everywhere where the functions in question are radial limits of f composed with ϕ. If $f\in L^1(T)$ and ϕ is inner, then $f(\phi)$ has the usual meaning of composition. But we can give other identifications which we need. Now $f=f_1+\bar{f}_2$ where $f_1,f_2\in H^p$ for $0<p<1$ [5]. So $f_1(\phi)$ and $f_2(\phi)$ are then the radial limits of $f_1(\phi(re^{it}))$ and $f_2(\phi(re^{it}))$, respectively. So we can think of $f(\phi(e^{it}))$ as the radial limit of the harmonic function $\tilde{f}(\phi(re^{it}))$ where $\tilde{f}$ is the harmonic extension of f to the disc D. So then $\lim_{r\to 1}\tilde{f}(\phi(re^{it}))=f(\phi(e^{it}))$ almost everywhere and $f(\phi)=f_1(\phi)+\overline{f_2(\phi)}$.

Suppose ϕ is an inner function having a fixed point $\alpha\in D$. Then it is not hard to see that the measure $P_\alpha dm$ is invariant under ϕ where $P_\alpha(t)=(1-|\alpha|^2)/|1-\bar{\alpha}e^{it}|^2$ the Poisson kernel and dm is Lebesgue measure on T.

Let

$$S_\alpha(z)=\frac{z+\alpha}{1+\bar{\alpha}z},\qquad \psi_\alpha(z)=S_{-\alpha}\circ\phi\circ S_\alpha(z).$$

Then $\psi_\alpha(0)=0$ and Lebesgue measure is invariant under ψ_α. Furthermore ergodicity and mixing of ψ with respect to $P_\alpha dm$ is equivalent to ergodicity and mixing of ψ_α with respect to dm.

We suppose now that ϕ is inner, $\phi(0)=0$ and $\phi(z)=zS(z)$ where $S(z)$ is a non-trivial inner function. Then $\phi_n(z)=\phi(\phi_{n-1}(z))=z\prod_{k=0}^{n-1}S(\phi_k(z))$. Schwarz's lemma implies $|\phi_n(z)|\leqq|\phi_{n-1}(z)|$ and from this and the form of ϕ_n we find that $\lim\phi_n(z)=0$ for all $z\in D$.

We need a result of this nature when ϕ does not have a fixed point in D. The result we state is due to Denjoy and Wolff and the version of the theorem stated is due to Heins [6].

THEOREM 1. *Let ϕ be an analytic map of D into itself having no fixed point in D. Then there is a unique β, $|\beta|=1$ such that*

$$\mathrm{Re}\left(\frac{\beta+\phi(z)}{\beta-\phi(z)}\right)\geqq\mathrm{Re}\left(\frac{\beta+z}{\beta-z}\right)\qquad \textit{for all } z\in D.$$

We can immediately deduce from the inequality in Theorem 1 along with Harnack's theorem that $\lim\phi_n(z)=\beta$ for all $z\in D$. And if ϕ is continuous at β, then $|\phi(\beta)|=1$. In what follows we will call the point β of Theorem 1 the Denjoy–Wolff point of ϕ.

The significance of the inequality in the theorem is that it gives us a way to identify the Denjoy–Wolff point from amongst all other "fixed points" of ϕ on T. The inequality implies that $\lim_{z\to\beta}\{(\beta-\phi(z))/(\beta-z)\}=\gamma$ as z approaches β in an angle and $|\gamma|\leqq 1$. In particular, if ϕ has a derivative at β, then $\phi'(\beta)=\gamma$ and $|\phi'(\beta)|\leqq 1$. See [4, p. 9] for further details on the Caratheodory derivative.

THEOREM 2. *If ϕ is inner with a fixed point $\alpha\in D$, and ϕ is not invertible, then ϕ is exact. That is, if B is the σ-ring of all Borel sets on T and $E\in\bigcap\phi^{-n}(B)$, then $m(E)$ is zero or one. In particular this shows ϕ is mixing of all orders and thus ergodic.*

PROOF. By writing $\psi(z)=S_{-\alpha}\circ\phi\circ S_\alpha(z)$ the exactness of ϕ is equivalent to the exactness of ψ, so we can assume $\phi(0)=0$. Let $E\in\bigcap\phi^{-n}(B)$. Then for every n we can find a set $E_n\in B$ so that $E=\phi^{-n}(E_n)$. Since Lebesgue measure is preserved by ϕ, we have $m(E)=m(E_n)$ for all n. Letting X_E and X_{E_n} be the characteristic function of E and E_n, respectively, we find that

$$\int X_E(t)P_z(t)dt=\int X_{E_n}(\phi_n)P_z(t)dt=\int X_{E_n}(t)P_{\phi_n(z)}(t)dt$$

for all $z\in D$ and all n. Since $\lim\phi_n(z)=0$, we see that $\lim P_{\phi_n(z)}(t)=1$ uniformly on T for all $z\in D$. Thus

$$\int X_E(t)P_z(t)dt = m(E) + \lim \int X_{E_n}(t)[P_{\phi_n(z)}(t) - 1]dt$$

$$= m(E) = \int X_E(t)dt.$$

Therefore $X_E(t)$ is constant almost everywhere and the theorem is proven.

COROLLARY 1. *Suppose ϕ is an inner funtion with a fixed point $\alpha \in D$. Then $P_\alpha dm$ is the unique absolutely continuous invariant probability measure if and only if there is no integer $n \geqq 1$ so that $\phi_n(z) = z$ for all $z \in D$.*

PROOF. We consider two cases, ϕ not invertible, and ϕ invertible. If ϕ is not invertible, we can use Theorem 2. We can assume $\alpha = 0$ and suppose gdm is invariant under ϕ where $g \in L^1(T)$. Then we have $\int fgdm = \int f(\phi_n)gdm$ and, therefore, $\lim \int f(\phi_n)gdm = \int f \int g$ for all $f \in C(T)$. Therefore, g is constant almost everywhere. Now suppose ϕ is 1-1. Then $\phi(z) = \lambda \cdot \{(z + \beta)/(1 + \bar{\beta}z)\}$ for some $\beta \in D$ and $|\lambda| = 1$. Since $\phi(\alpha) = \alpha$ for some $\alpha \in D$ we have $\psi(z) = S_{-\alpha} \circ \phi \circ S_\alpha(z) = \gamma z$ for some constant γ, $|\gamma| = 1$. Now γ is not an n-th root of unity if and only if $\phi_n(z) \neq z$ for some $z \in D$. But Lebesgue measure is the unique absolutely continuous invariant probability measure for $\psi(z)$ if and only if γ is not a root of unity. This says $P_\alpha dm$ is the unique absolutely continuous invariant probability measure for ϕ if and only if there is no $n \geqq 1$ such that $\phi_n(z) = z$ for all $z \in D$.

We can transfer results from the circle to the line by a conformal map. That is, let $w = \Phi(z) = i\{(1 + z)/(1 - z)\}$ be the conformal map of D onto the upper half plane of $\mathbf{C}$, the complex plane. Φ carries T onto R with $\Phi^{-1}(w) = (w - i)/(w + i)$. If ϕ is inner on D, then $\psi = \Phi \circ \phi \circ \Phi^{-1}$ is inner on the upper half plane and conversely where we mean that ψ is an analytic map of the upper half plane onto itself and sending R into R. Φ carries the Poisson measures $P_\alpha dt$ on T to the Cauchy measure $Q_w dx$ on the line where

$$Q_w(x) = \frac{1}{\pi} \frac{b}{(x + a)^2 + b^2}, \qquad w = \Phi(z) = a + ib,$$

and dx is Lebesgue measure on the line. Furthermore, Φ carries $dt/(1 - \cos t)$ on T to Lebesgue measure dx on R.

It is clear that if ϕ is ergodic with respect to dt on T, then it is ergodic with respect to $dt/(1 - \cos t)$ and conversely, and then ψ is ergodic with respect to dx on R. Similarly ϕ being exact with respect to $P_\alpha dt$ on T implies ψ is exact with respect to $Q_w dx$ on R, and weak mixing is also preserved.

The referee pointed out an error in the statement of Theorem 2 and suggested the strengthened statement of Corollary 1. He also pointed out that Theorem 2 transferred to the upper half plane gives a proof of a generalization of a result of Kemperman for meromorphic functions which was announced in [8]. Kemperman discusses the function

$$g(z) = A + Bz - \sum p_k \left[\frac{1}{z - c_k} + \frac{1}{c_k}\right]$$

where A, B, p_k, c_k are real constants such that $B \geqq 0$, $p_k > 0$, $\Sigma p_k / c_k^2 < \infty$ and c_k has no finite accumulation point. He assumes that $B < 1$ and $|g'(x)| > 1$ for all $-\infty < x < \infty$. He then states that g has a unique invariant probability measure equivalent to Lebesgue measure and he conjectures that g is exact and Bernoulli.

$g(z)$ is inner in the upper half plane and since $|g'(x)| > 1$ Theorem 1 implies g must have a fixed point in the upper half plane. Therefore, Theorem 2 implies exactness. The problem of whether g is Bernoulli will be discussed in another paper.

We now consider the case when ϕ has no fixed points in D. We will show ϕ_n is weak mixing for all n, where by weak mixing we mean that there are only trivial L^∞ solutions to $f(\phi) = \lambda f$ where $|\lambda| = 1$.

We make one final normalization. If $\beta \in T$ and is the Denjoy–Wolff point for ϕ, letting $\tilde{\phi}(z) = \bar{\beta}\phi(\beta z)$, we see that $\tilde{\phi}$ has $z = 1$ as its Denjoy–Wolff point. Further ϕ is weak mixing if and only if $\tilde{\phi}$ is.

THEOREM 3. *Suppose ϕ is inner and has no fixed points in D. If $\Sigma(1 - |\phi_n(0)|) = \infty$ then ϕ_m is weak mixing for all $m \geqq 1$.*

PROOF. First we consider the case $\lambda = 1$. Suppose there is an $f \in L^\infty$ such that $f(\phi) = f$. Now $f = g + \bar{h}$ where $g, h \in H^1$. Since f is invariant we must have $g(\phi) = g + c$, $h(\phi) = h - \bar{c}$, c a constant.

(a) Suppose $c = 0$. Then $g(\phi(z)) = g(z)$ for all $z \in D$ and in particular $g(\phi_n(0)) = g(0)$ for all n. The "Blaschke" condition of the theorem implies g is constant, similarly so is h and, therefore, so is f.

(b) Suppose $c \neq 0$. Then let $G(z) = \exp\{(2\pi i/c)g(z)\}$. Since $g \in H^1$, we have $G \in N$ and $g(\phi_n(0)) = G(0)$. So again the "Blaschke" condition implies G is constant and, therefore, so is g. Similarly so is h and, therefore, f is constant.

Now suppose $\lambda \neq 1$ but $|\lambda| = 1$. We assume there is an $f \in L^\infty$ such that $f(\phi) = \lambda f$. Again as above write $f = g + \bar{h}$, $g, h \in H^1$. The functional equation for f implies $g(\phi) = \lambda g + c$ and a similar functional equation for h. Let $b = c/(\lambda - 1)$ and $G(z) = g(z) + b$. Then $G(\phi) = \lambda G$. Since ϕ is ergodic, $|G|$ is constant almost everywhere, and so $G(z)$ is in H^∞. If $G(\alpha)=0$ for some $\alpha \in D$, then $G\phi_n(\alpha)=0$ for all $n \geqq 1$. Aaronson [1] showed that if $\Sigma(1-|\phi_n(0)|)=\infty$, then $\Sigma(1-|\phi_n(z)|)=\infty$ for all $z \in D$. This then implies that G is constant and we are finished. So we can assume $G(z)\neq 0$ for all $z \in D$. Letting $U(z)=G(z)^{2\pi i/\log\lambda}$, we see $U(z)\in H^\infty$ and $U(\phi(z))=U(z)$. So U is constant and, therefore, G is and this says ϕ is weak mixing. In order to show ϕ_m is weak mixing for all m, we show the "Blaschke" condition on $\phi_n(0)$ implies

$$\Sigma(1-|\phi_{rn+s}(0)|)=\infty \qquad \text{for } 1\leqq r, \quad 0\leqq s\leqq r-1.$$

Suppose that $\Sigma(1-|\phi_{rn+s}(0)|)<\infty$ for some r, s, $0\leqq s\leqq r-1$. Let $B(z)$ be the Blaschke product with $\{\phi_{rn+s}(0)\}$ as its zeros. Then $B(\phi(z))$ has $\{\phi_{rn+s-1}(0)\}$ as zeros so that $\Sigma(1-|\phi_{rn+s-1}(0)|)<\infty$. By induction then $\Sigma(1-|\phi_{rn+s}(0)|)<\infty$ for $0\leqq s\leqq r-1$, and, therefore, $\Sigma(1-|\phi_n(0)|)<\infty$, a contradiction. So this gives the proof of the theorem.

We make some remarks on the converse to Theorem 3. If $\Sigma(1-|\phi_n(0)|)<\infty$ then we can form the Blaschke product

$$B(z)=z\prod_{n=1}^{\infty}\frac{\bar{\alpha}_n}{|\alpha_n|}\,\frac{\alpha_n-z}{1-\bar{\alpha}_n z}$$

where $\alpha_n=\phi_n(0)$.

It is not hard to show that ϕ is not ergodic if and only if there is a $z\in D$ and a subsequence n_k of integers such that $\lim B(\phi_{n_k}(z))\neq 0$. We have not been able to show this and we leave it as a problem.

In order to show ϕ is not ergodic it is enough to find an eigenvalue λ, $|\lambda|<1$ and a measurable "eigenfunction" f such that $f(\phi)=\lambda f$. By taking absolute values we can assume $f\geqq 0$ and $0<\lambda<1$. A solution to this equation leads immediately to one for $g(\phi)=g$ as follows. Let

$$\begin{aligned} g(t) &= \sin\frac{2\pi}{\log\lambda}\log f(t) \qquad && \text{when } f(t)\neq 0\\ &= 0 && \text{when } f(t)=0. \end{aligned}$$

Then g is a bounded and measurable non-trivial solution to $g(\phi)=g$, so that ϕ is not ergodic. The existence of an eigenvalue follows in certain cases from the following theorem which may be found in a paper by Karlin and MacGregor [7, p. 139].

THEOREM 4. *Let ϕ be analytic in a neighborhood of a fixed point β with $\phi'(\beta)=\gamma$, $0<|\gamma|<1$. Let G be a connected domain containing β and such that*

$\phi(G) \subseteq G$. Furthermore suppose $z \in G$ implies $\phi'(z) \neq 0$ and $\phi_n(z) \to \beta$. Then uniformly in a neighborhood of β we have $\lim(\beta - \phi_n(z))/\gamma^n = A(z)$.

The referee has kindly provided the statement and proof of the following proposition.

PROPOSITION 1. *Let ϕ be an inner function with Denjoy–Wolff point $\beta \in T$. Assume that $\phi'(\beta) = \gamma$, $|\gamma| < 1$, and ϕ has an analytic extension around β. Then ϕ is not ergodic. We note $\phi'(\beta) \neq 0$ since $\beta \in T$.*

PROOF. Since ϕ is analytic in a neighborhood of β there is an $\varepsilon > 0$ and $\alpha < 1$ such that (1) $\phi'(z) \neq 0$, (2) $|\phi(z) - \beta| \leqq \alpha |z - \beta|$ for $|z - \beta| \leqq \varepsilon$ and $G = \{z : |z - \beta| \leqq \varepsilon\}$ satisfies the conditions of Theorem 4. Therefore

$$\frac{\beta - \phi_n(z)}{\gamma^n} \to A(z) \qquad \text{for } |z - \beta| < \varepsilon.$$

Let $A = \{t \in T : \phi_n(t) \to \beta\}$. Clearly A is ϕ-invariant and $\{|z - \beta| < \varepsilon\} \cap T \subseteq A$, and $m(\{|z - \beta| < \varepsilon\} \cap T) > 0$. If ϕ were ergodic, then $A = T$ almost everywhere. Now $A(\phi(t)) = \gamma A(t)$ for $t \in T$. Then by the remarks preceding Theorem 4 we get a non-trivial solution to $f(\phi) = f$, a contradiction.

In particular, if ϕ is a finite Blaschke product with Denjoy–Wolff point $\beta \in T$, and $|\phi'(\beta)| < 1$, then the proposition implies ϕ is not ergodic. The condition $|\phi'(\beta)| < 1$ makes the series $\Sigma(1 - |\phi_n(0)|)$ converge geometrically. It seems to be a reasonable conjecture that the geometric convergence of $\Sigma(1 - |\phi_n(0)|)$ implies ϕ is not ergodic. However, it isn't hard to give examples of non-ergodic ϕ with

$$\lim \frac{1 - |\phi_n(0)|}{1 - |\phi_{n+1}(0)|} = 1.$$

One is led to ask whether $\Sigma(1 - |\phi_n(0)|) < \infty$ implies non-ergodicity. We leave this as an open problem. When ϕ has a fixed point $\alpha \in D$, we know $P_\alpha dm$ is the unique absolutely continuous invariant probaiblity measure for ϕ. When the Denjoy–Wolff point is on T, we have

THEOREM 5. *If ϕ is inner with Denjoy–Wolff point $\beta \in T$, then ϕ has no finite non-trivial absolutely continuous invariant measure.*

PROOF. By hypothesis $\lim \phi_n(z) = \beta$ for all $z \in D$, $|\beta| = 1$. If g is a polynomial, we have $\lim \int g(\phi_n) P_z dt = \lim g(\phi_n(z)) = g(\beta)$. Therefore, $\lim \int g(\phi_n) h dt = g(\beta) \int h dt$ for all $g \in C(T)$ and h is in the linear span of $\{P_z, z \in D\}$. But the linear span of the Poisson kernels $P_z, z \in D$ is dense in L^p,

$1 \leqq p < \infty$. Therefore, $\lim \int g(\phi_n) h dt = g(\beta) \int h dt$ for all $g \in C(T)$ and $h \in L^1(T)$. Now if $d\mu = f dm$ is a finite invariant measure, we get

$$\lim \int g(\phi_n) f dm = \int g f dm = g(\beta) \int f dm \qquad \text{for all } g \in C(T).$$

So $d\mu$ is the mass point at β, a contradiction.

However ϕ can have σ-finite absolutely continuous invariant measures. In fact if ϕ is inner and $\overline{\phi(z)} = \phi(\bar{z})$ and $z = 1$ is the Denjoy–Wolff point of ϕ and $\phi'(1) = 1$, then $dt/(1 - \cos t)$ is invariant under ϕ. See [1] for futher details.

Now if ϕ is ergodic with respect to an invariant absolutely continuous σ-finite measure μ, then μ is unique up to constant multiples. Since this is probably a well-known fact about ergodic measures, we just outline a proof.

The problem is reduced to the case where the measure is finite by the induced transformation of Kakutani. See [2] for details on the induced transformation. Then an application of the ergodic theorem gives the result.

Adler and Weiss [2] studied the Boole transformation $\psi(t) = t - 1/t$ on R and showed dt is invariant under ψ and ψ is totally ergodic.

We use Theorem 3 to show, in fact, that ψ_n is weak mixing for all n. In fact, let $\psi_\lambda(t) = \lambda(t - 1/t)$. Then

$$\phi(z) = \Phi^{-1} \circ \psi_\lambda \circ \Phi(z) = \frac{z^2 + \alpha}{\alpha z^2 + 1} \qquad \text{where } \alpha = \frac{2\lambda - 1}{2\lambda + 1}.$$

When $0 < \lambda < 1$, then clearly $\psi_\lambda(z)$ has a fixed point in the upper half plane so that ψ_λ is exact with a finite absolutely continuous invariant measure. If $\lambda > 1$, then $z = 1$ is the Denjoy–Wolff point for ϕ_λ and $|\phi'_\lambda(1)| < 1$. Therefore, in this case, ϕ_λ and so ψ_λ are not ergodic. When $\lambda = 1$, $z = 1$ is again the Denjoy–Wolff point for ϕ_λ and $\phi'_\lambda(1) = 1$. Now $i = \Phi(0)$ so that $\psi_n(i) = \Phi \circ \phi_n \circ \Phi^{-1}(i) = \Phi(\phi_n(0))$. The condition $\Sigma_{n=0}^\infty (1 - |\phi_n(0)|) = \infty$ is equivalent to $\Sigma_{n=0}^\infty \operatorname{Im} \psi_n(i)/(1 + |\psi_n(i)|^2) = \infty$ and this is enough to show ψ is totally weak mixing.

THEOREM 6 (Adler–Weiss). *The mapping $\psi(f) = t - 1/t$ is totally weak mixing.*

PROOF. Let $\alpha_n = \psi_n(i)$. Then $\alpha_n = i\beta_n$, where $\beta_n > 0$ and $\lim \beta_n = \infty$. This is because $z = 1$ is the Denjoy–Wolff point for ϕ and so ∞ is the Denjoy–Wolff point for ψ. Now $\Sigma \beta_n/(1 + \beta_n^2) = \Sigma 1/\beta_{n+1}$ because $\beta_{n+1} = \beta_n + 1/\beta_n$. But then $\Sigma_1^\infty 1/\beta_k = \lim \beta_{n+1} - \beta_1 = \infty$. So the theorem is proven.

We give one more example. Let $\phi(z) = \Pi_1^N (z + \alpha_k)/(1 + \alpha_k z)$ where the α_k are real. When N is even we have the induced

$$\psi(z)=\frac{z}{\omega}+\frac{c_1z^{N-2}+\cdots+c_r/z}{\omega z^{N-1}+b_1z^{N-3}+\cdots+b_sz}$$

and when N is odd

$$\psi(z)=\frac{z}{\omega}+\frac{c_1z^{N-2}+\cdots+c_r}{\omega z^{N-1}+\cdots+b_s}.$$

Here $\omega=\Sigma_1^N(1-\alpha_k)/(1+\alpha_k)=\phi'(1)$. Since $z=1$ is a fixed point for ϕ, when $\omega\leqq 1$, $z=1$ is the Denjoy–Wolff point for ϕ and $z=\infty$ is the Denjoy–Wolff point for ψ. When $\omega<1$, ϕ, and therefore ψ, are not ergodic. When $\omega=1$ we show ϕ and so ψ is ergodic. This case was studied in [10] where only ergodicity was proven. We have when N is even or odd that $\psi_n(i)=i\beta_n$, $\beta_n>0$, and $\lim\beta_n=\infty$. Now

$$\beta_{n+1}=\beta_n+\frac{c_1\beta_n+O\left(\frac{1}{\beta_n}\right)}{\beta_n^2+1+O\left(\frac{1}{\beta_n}\right)}.$$

Then $\Sigma\beta_n/(1+\beta_n^2)\approx\Sigma 1/\beta_n=\infty$ and so we have total weak mixing.

Acknowledgement

We would like to express our thanks to the referee for pointing out references [8] and [10], for the statement and proof of Proposition 1, and for providing the correct statement to Theorem 2 and Corollary 1.

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